JOURNAL OF MULTIVARIATE ANALYSIS 31, 258–265 (1989)

Weak Convergence of the U-Statistic and Weak Invariance of the One-Sample Rank Order Statistic for Markov Processes and ARMA Models

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Communicated by the Editors

Harel and Puri (1989, *J. Multivariate Anal.* **29**) studied the asymptotic behavior of the U-statistic and the one-sample rank order statistic for nonstationary absolutely regular processes. In this note, we present some applications of these results for Markov processes as well as ARMA processes. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let X_{ni}, $1 \leqslant i \leqslant n$, $n \geqslant 1$ be real-valued r.v.'s (random variables) with continuous d.f.'s (distribution functions) $F_{ni}(x)$, $x \in \mathbb{R}$, and let $\mathbf{F}_n$ denote the d.f. of the $\mathbb{R}^n$-valued r.v. $(X_{n1}, ..., X_{nn})$.

Consider the U-statistic

$$U(\mathbf{F}_n) = \binom{n}{k}^{-1} \sum_{(i)}^{(n)} g(X_{ni_1}, ..., X_{ni_k}), \qquad n \geqslant k \geqslant 1, \tag{1.1}$$

where the summation $\sum_{(i)}^{(n)}$ extends over all possible $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and g: $\mathbb{R}^k \to \mathbb{R}$ is a Borel measurable function which is symmetric in its $k(\geqslant 1)$ arguments.

Received December 21, 1988.

AMS 1980 subject classifications: Primary 60B10, 60F05.

Key words and phrases: U-statistic, rank order statistic, absolute regularity, Markov processes, ARMA processes.

* Research supported by the Office of Naval Research Contract N00014-85-K-0648.

0047-259X/89 $3.00

Consider also the 1-sample rank order statistic $\mathcal{S}_{n,m}$

$$\mathcal{S}_{n,m} = \sum_{i=1}^{m} c_{ni} s(X_{ni})\, J\left(\frac{R_{n,m,i}}{m+1}\right), \qquad n \geqslant m \geqslant 1, \tag{1.2}$$

where J is a score function, $s(x) = \operatorname{sgn}(x)$ and the c_{ni} are regression constants generated by a continuous function $h(x)$ on $[0, 1]$,

$$\begin{aligned} c_{ni} &= h\left(\frac{i}{n}\right), \ 1 \leqslant i \leqslant n, \ n \geqslant 1, \\ R_{n,m,i} &= \sum_{j=1}^{m} I_{[|X_{nj}| \leqslant |X_{ni}|]}, \ 1 \leqslant i \leqslant m \leqslant n, \end{aligned} \tag{1.3}$$

where $I_{[\ ]}$ denotes the indicator function.

In Harel and Puri [3], we studied the asymptotic behavior of the statistics $U(\mathbf{F}_n)$ and $\mathcal{S}_{n,m}$ when the underlying r.v.'s are nonstationary, absolutely regular with rates

$$\beta(m) = O(m^{-(2+\delta)/\delta}) \qquad \text{for some} \quad \delta > 0 \tag{1.4}$$

in the case of the U-statistic (1.1), and

$$\beta(m) = O(m^{-4}) \tag{1.5}$$

in the case of the 1-sample rank statistic (1.2). In this paper, we provide applications of some of the results of Harel and Puri [3] for some Markov processes as well as ARMA processes.

For the ease of convenience and easy reference we state the two main results from Harel and Puri [3].

THEOREM 1.1 (Convergence of the U-statistic). *Let $F_{n,i,j}$ be the d.f. of (X_{ni}, X_{nj}), $1 \leqslant i < j \leqslant n$. Suppose that in addition to the assumption* (1.4), *the assumptions* (2.6) *and* (2.7) *of Harel and Puri* [3] *are satisfied. Furthermore, assume that for any $i, j \in \mathbb{N}^*$, with $i < j$, there exists a continuous d.f. F_{ij} on $\mathbb{R}^2$ with continuous marginals F_i^* and F_j^* such that*

$$\lim_{n \to \infty} \max_{1 \leqslant i < j \leqslant n} |F_{n,i,j}(x_1, x_2) - F_{ij}(x_1, x_2)| = 0 \qquad \textit{for all} \quad (x_1, x_2) \in \mathbb{R}^2 \tag{1.6}$$

$$F_{ij} = F_{1,j-i+1} \qquad \textit{for all} \quad i < j \tag{1.7}$$

and

$$\int_{\mathbb{R}^k} |g(x_1, ..., x_k)|^r \prod_{j=1}^{k} dF(x_j) \leqslant M_0 < \infty, \tag{1.8}$$

where M_0 is some constant >0 and $F=F_i^*$ for all $i\in\mathbb{N}^*$ and g is right continuous and has left-hand limits (r.c.l.l.) or left continuous and has right-hand limits (r.c.r.l.). Then $n^{1/2}(U(\mathbf{F}_n)-\theta(\mathbf{F}_n))$ converges in law to the normal distribution with mean 0 and variance $k^2\sigma^2$ if $\sigma^2>0$, where $\theta(\mathbf{F}_n)= n^{-[k]}\sum_{(i_1,\ldots,i_k)\in\mathbf{I}_{0,n}}\int_{\mathbb{R}^k} g(x_1,\ldots,x_k)\,dF_{ni_1}\cdots dF_{ni_k}$, where $\mathbf{I}_{0,n}=\{i_1,\ldots,i_k);\ 1\leqslant i_1\neq\cdots\neq i_k\leqslant n\}$,

$$\begin{aligned}\sigma^2=&\left[\int_{\mathbb{R}^k} g^2(x_1,\ldots,x_k)\prod_{l=1}^{k} dF(x_l)-\theta^2(F)\right]\\ &+2\sum_{i=2}^{\infty}\left[\int_{\mathbb{R}^{2k}} g(x_1,\ldots,x_k)\,g(x_{k+1},\ldots,x_{2k})\right.\\ &\times dF_{1i}(x_1,x_{k+1})\prod_{l=2}^{k} dF(x_l)^2\\ &\left.\times\prod_{p=k+2}^{k} dF(x_p)-\theta^2(F)\right]<\infty,\end{aligned}\tag{1.9}$$

where $\theta(F)=\int_{\mathbb{R}^k} g(x_1,\ldots,x_k)\prod_{l=1}^{k} dF(x_l)$.

We now assume that $F_{ni}=F_n$ for any i $(1\leqslant i\leqslant n)$. For any real number x, define $H_n(|x|)$ as $H_n(|x|)=F_n(|x|)-F_n(-|x|)$. Let F be a d.f. on $\mathbb{R}$ and define the d.f. H on $\mathbb{R}^+$ by $H(|x|)=F(|x|)-F(-|x|)$.

For a score function $J(u)$ which is square integrable put

$$\mu_n=\mu_J(F_n)=\int_{\mathbb{R}} s(x)\,J(H_n(|x|)\,dF_n(x).\tag{1.10}$$

For h defined in (1.3) put

$$\mu_{n,m}=\mu_n\sum_{i=1}^{m} c_{ni}=\mu_n\sum_{i=1}^{m} h\left(\frac{i}{n}\right),\qquad m\leqslant n.\tag{1.11}$$

For any sequence of d.f.'s $\{F_{1i};\ i\geqslant 2\}$ on $\mathbb{R}^2$ with marginals F we denote

$$\begin{aligned}\sigma^2(F_{1i}\})=\lim_{n\to\infty}&\left\{\int_{\mathbb{R}} g^2(x)\,dF(x)\right.\\ &\left.+2\sum_{l=2}^{n}\int_{\mathbb{R}^2} g(x)\,g(y)\,dF_{1l}(x,y)\right\}\end{aligned}\tag{1.12}$$

if the limit exists, where

$$\begin{aligned}g(x)=&\int_{\mathbb{R}} s(y)\,\{I_{[|x|\leqslant|y|]}-H(|y|)\}\,J'(H(|y|)\,dF(y)\\ &+s(x)\,J(H(|x|))-\int_{\mathbb{R}} s(y)\,J(H(|y|)\,dF(y).\end{aligned}$$

For every $n \geqslant 1$, let

$$V_n(t) = \begin{cases} 0, & \text{for} \quad t=0 \\ (\mathscr{S}_{n,k} - \mu_{n,k})/\sigma^{1/2}, & \text{for} \quad t=k/n,\ k=1, ..., n \\ \text{linearly interpolated,} & \text{for} \quad t \in [k-1/n, k/n],\ k=1, ..., n, \end{cases} \tag{1.13}$$

where σ is the positive constant defined in (1.12).

The proces $V_n(t) = \{V_n(t),\ 0 \leqslant t \leqslant 1\}$ belongs to the space C_1 of all continuous functions on $[0, 1]$ with which we associate the usual uniform metric. Then, we have

THEOREM 1.2 (Convergence of the rank order statistic). *Suppose the sequence $\{X_{ni}\}$ is absolutely regular with rate* (1.5) *and the sequence $\{F_n\}$ satisfies the conditions* (1.6) *and* (1.7) *of Theorem* 1.1. *Let J be a score function having a bounded second derivative. If $\sigma^2(\{F_{1i}\})$ defined by* (1.12) *is strictly positive, then V_n defined in* (1.13) *converges weakly in the uniform topology on C_1 to the process $V_0 = \{V_0(t), 0 \leqslant t \leqslant 1\}$, where*

$$V_0(t) = \int_0^t h(u)\, dW(u), \qquad 0 \leqslant t \leqslant 1, \tag{1.14}$$

and $W = \{W(t),\ 0 \leqslant t \leqslant 1\}$ is a standard Brownian motion process, and $\sigma^2(\{F_{1i}\} < \infty$.

2. Applications to Markov Processes and Arma Processes

2.1. *Markov Processes*

Consider a sequence $\{X_{ni};\ i \in \mathbb{Z}\}$ *of* $\mathbb{R}$-valued processes such that for all $n \in \mathbb{N}^*$, $\{X_{ni}\}$ is a Markov process with stationary transition probabilities $P_n(x; A)$, where $A \in \mathscr{B}$, $\mathscr{B}$ is the Borel σ-field of $\mathbb{R}$, and $x \in \mathbb{R}$.

Recall that the Markov process is *geometrically ergodic* if it is ergodic and if there exists $0 < \rho_n < 1$ such that

$$\|P_n^m(x;\cdot) - \mu_n(\cdot)\| = O(\rho_n^m) \qquad \text{for all} \quad \text{a.s. } x \in \mathbb{R},$$

where $\|\cdot\|$ denotes the norm of total variation. (ρ_n is called the rate) and P_n^m is the m-step transition probability.

THEOREM 2.1. *Let $\{X_{ni};\ i \in \mathbb{Z}\}$ be a Markov process such that for every $n \in \mathbb{N}^*$, $\{X_{ni}\}$ is either* (a) *aperiodic, Harris recurrent, and geometrically* *ergodic with rates* $0<\rho_n<\rho_0$, $\rho_0\in(0,1)$ *or* (b) *aperiodic and Doeblin recurrent.*

Suppose there exists a probability measure μ_0 *on* $\mathbb{R}$ *and a transition probability* P_0 *such that*

$$\mu_n((-\infty,x])\to\mu_0((-\infty,x]) \qquad as \quad n\to\infty \tag{2.1}$$

for all $x\in\mathbb{R}$;

$$P_n(x;(-\infty,y])\to P_0(x;(-\infty,y]) \qquad as \quad n\to\infty \tag{2.2}$$

for all $(x,y)\in\mathbb{R}^2$. *Then for a function g satisfying the conditions of Theorem* 1.1 *and for a U-statistic defined in* (1.1), *the conclusion of Theorem* 1.1 *holds. Also for a score function J having a second bounded derivative and the process* V_n *defined in* (1.13), *the conclusion of Theorem* 1.2 *holds.*

Proof. (i) Suppose (a) holds. From Theorems 1.1 and 1.2, we have to show that the conditions (1.6) and (1.7) are verified and the sequence $\{X_{ni}\}$ *is absolutely regular with the rate* (1.4) *or* (1.5).

From Nummelin and Tuominen [5], a Markov process which is aperiodic, Harris recurrent, and gemetrically ergodic with rates ρ_n, satisfies

$$\int_{\mathbb{R}} \|P_n^m(x;\cdot)-\mu_n(\cdot)\|\,\mu_n(dx)=O(\rho_n^m).$$

From Proposition 1 of Davydov [2] we have

$$\beta(m)=\sup_{n\in\mathbb{N}^*}\int_{\mathbb{R}} \|P_n^m(x;\cdot)-\mu_n(\cdot)\|\,\mu_n(dx)$$

and we deduce that the sequence $\{X_{ni}\}$ is absolutely regular with a geometrical rate ρ_0 which implies (1.4) or (1.5).

We now show that (1.6) is satisfied. For any $n\geqslant 1$, it can easily be seen that the d.f. $F_{n,i,j}$ has the same marginals F_n, say. For any $m\geqslant 1$, let P_0^m be the m-step transition probability of P_0.

Let G_m be the d.f. associated with the probability measure Q_m, where

$$Q_m(A_1\times A_2)=\int_{A_1}\int_{A_2} P_0^m(x;dy)\,\mu_0(dx) \qquad \forall(A_1,A_2)\in\mathscr{B}^2.$$

For any $(i,j)\in(\mathbb{N}^*)^2$ with $i<j$, we consider the d.f. F_{ij} defined by $F_{ij}=G_{j-i}$. From the definition of F_{ij}, it is clear that the marginal d.f.'s of F_{ij} are identical. We denote them by F.

Let $(x, y) \in \mathbb{R}^2$ be fixed, and note the inequality

$$\begin{aligned}|F_{n,i,j}(x, y) - F_{ij}(x, y)| \leqslant & \, |F_{n,i,j}(x, y) - F_n(x)\, F_n(y)| \\ & + |F_n(x) - F(x)|\, F_n(y) \\ & + F(x)\, |F_n(y) - F(y)| \\ & + |F_{ij}(x, y) - F(x)\, F(y)|. \end{aligned} \tag{2.3}$$

As the sequence $\{X_{ni}\}$ is absolutely regular with the geometrical rate ρ_0, it is also strong mixing with the same rate and, from the definition of strong mixing, we deduce that $\forall \varepsilon > 0\ \exists m_0 \in \mathbb{N}^*$ such that $\forall (i, j) \in (\mathbb{N}^*)^2$ with $i < j$ and $j - i \geqslant m_0$,

$$|F_{n,i,j}(x, y) - F_n(x)\, F_n(y)| < A\rho_0^m \leqslant \varepsilon/4, \tag{2.4}$$

where A is some positive constant, and from (2.1) and (2.2), we also obtain

$$|F_{ij}(x, y) - F(x)\, F(y)| < \varepsilon/4. \tag{2.5}$$

Now from conditions (2.1) and (2.2), we also deduce that $\forall \varepsilon > 0\ \exists n_0$ such that $\forall n \geqslant n_0$ and $\forall (i, j) \in (\mathbb{N}^*)^2$ with $i < j$ and $j - i < m_0$,

$$|F_{n,i,j}((x, y) - F_{ij}(x, y)| < \varepsilon \tag{2.6}$$

$$|F_n(x) - F(x)| < \varepsilon/4 \tag{2.7}$$

$$|F_n(y) - F(y)| < \varepsilon/4. \tag{2.8}$$

From (2.3)–(2.5) and (2.7), (2.8) we deduce that $\forall n \geqslant n_0$, $\forall (i, j) \in (\mathbb{N}^*)^2$ with $i < j$ and $j - i \geqslant m_0$,

$$|F_{n,i,j}(x, y) - F_{ij}(x, y)| < \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon. \tag{2.9}$$

(2.6) and (2.9) yield (1.6). As the d.f. F_{ij} has the same marginals F, we obtain (1.7).

(ii) Suppose (b) holds. From Davydov [2], a Markov process is Doeblin recurrent and aperiodic is geometrically φ-mixing. This implies that $\{X_{ni}\}$ is also absolutely regular with a geometrical rate.

EXAMPLE 2.1. Consider the process $\{X_{ni};\ i \in \mathbb{Z}\}$, where $X_{n,i+1} = a_1^{(n)} X_{ni} + a_2^{(n)} X_{ni} \varepsilon_{i+1} + a_3^{(n)} \varepsilon_{i+1} + a_4^{(n)} \varepsilon_{i+1}^2 + a_5^{(n)}$ where the a's are real numbers and $\{\varepsilon_i; i \in \mathbb{Z}\}$ is a white noise with strictly positive density. Then Mokkadem [4] has shown that if $(a_1^{(n)})^2 + (a_2^{(n)})^2\, E(\varepsilon_1^2) < 1$ and $E(\varepsilon_1^4) < \infty$, then the process $\{X_{ni}; i \in \mathbb{Z}\}$ is geometrically ergodic.

If we have

$$\forall j \in \{1, ..., 5\} \qquad \exists a_j \in \mathbb{R}$$

such that

$$\lim_{n \to \infty} a_j^{(n)} = a_j$$

and

$$a_1^2 + a_2^2 E(\varepsilon_1^2) < 1$$

then the conditions (2.1) and (2.2) are satisfied and we can apply Theorem 2.1.

EXAMPLE 2.2. Consider the process $\{X_{ni}; i \in \mathbb{Z}\}$, where $X_{n,i+1} = f_n(X_{ni}) + \varepsilon_{i+1}$ where the ε's are independent and identically distributed random variables with strictly positive density and f_n: $\mathbb{R} \to \mathbb{R}$ is bounded and continuous. (This model was studied by Collomb and Doukhan [1].) It is easy to check that this model is Doeblin recurrent and aperiodic and we deduce that $\{X_{ni}; i \in \mathbb{Z}\}$ is geometrically φ-mixing and if f_n converges simply to a bounded and continuous function f_0, we can apply Theorem 2.1.

2.2. *ARMA Processes*

Consider a sequence of ARMA processes $\{X_{ni}; i \in \mathbb{Z}\}$, $n \in \mathbb{N}^*$,

$$X_{n,i+1} = \tilde{a}_1^{(n)} X_{ni} + \tilde{a}_2^{(n)} \varepsilon_{i+1} + \varepsilon_i \tag{2.10}$$

where $\{\varepsilon_i;\ i \in \mathbb{Z}\}$ is a sequence of independent random variables such that $E(\varepsilon_i) = 0$.

THEOREM 2.2. *Let $\{X_{ni}; i \in \mathbb{Z}\}$ be a sequence of ARMA processes given by* (2.10). *Suppose $\{X_{ni}\}$ satisfies the conditions*:

$\{\varepsilon_i;\ i \in \mathbb{Z}\}$ is a sequence of independent and identically distributed random variables with strictly positive density; (2.11)

$$\exists (\tilde{a}_1, \tilde{a}_2) \in (-1, 1) \times \mathbb{R} \tag{2.12}$$

such that

$$\lim_{n \to \infty} \tilde{a}_1^{(n)} = \tilde{a}_1 \qquad \textit{and} \qquad \lim_{n \to \infty} \tilde{a}_2^{(n)} = \tilde{a}_2.$$

Then for a function g satisfying the conditions of Theorem 1.1 *and for a U-statistic defined in* (1.1) *the conclusions of Theorem* 1.1 *hold.*

Also for a score function J having a second bounded derivative and for the process V_n defined in (1.13) *the conclusions of Theorem* 1.2 *hold.*

Proof. Particular case of Theorem 2.1.

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